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Spectral statistics in chaotic systems with a point interaction

Martin Sieber

Max-Planck-Institut für Physik komplexer Systeme, Nöthnitzer Str. 38, 01187 Dresden, Germany

E-mail: sieber@mpipks-dresden.mpg.de

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Abstract. We consider quantum systems with a chaotic classical limit that are perturbed by a point-like scatterer. The spectral form factor $K(\tau)$ for these systems is evaluated semiclassically in terms of periodic and diffractive orbits. It is shown for order τ^2 and τ^3 that off-diagonal contributions to the form factor which involve diffractive orbits cancel exactly the diagonal contributions from diffractive orbits, implying that the perturbation by the scatterer does not change the spectral statistic. We further show that parametric spectral statistics for these systems are universal for small changes of the strength of the scatterer.

1. Introduction

Semiclassical theories for spectral statistics have been developed [1–3] to find an explanation for the observed universality in energy spectra of quantum systems with a chaotic classical limit, i.e. the agreement of correlations in energy spectra with those between eigenvalues of random matrices [4]. They are based on semiclassical trace formulae that approximate the density of states in terms of classical trajectories [5]. It has been shown by these theories that in the asymptotic limit of long-range correlations two-point correlation functions do coincide with those of random matrix theory [2, 3]. These results are based on mean properties of periodic orbits [1]. To go beyond the leading asymptotic term requires information concerning correlations between periodic orbits that is presently not available [6].

One of the expectations, on the basis of the random matrix hypothesis [7], is that a perturbation of a chaotic system should not change the statistical distribution of the energy levels of the quantum system, if it does not change the chaotic nature of the classical motion. In this paper we investigate, at the level of the semiclassical approximation, whether the perturbation by a point-like scatterer has this property. One argument in favour of this invariance is that the semiclassical approximation for the density of states is not changed in leading order of \hbar for this perturbation. The influence of the scatterer is described semiclassically by a certain class of trajectories, so-called diffractive orbits that start from the scatterer and return to it. They contribute to the density of states in higher order of \hbar than the leading-order contribution from periodic orbits.

This paper is motivated by the observation in [8] that a scatterer could nevertheless have an influence on the spectral statistics. When spectral correlation functions are calculated by using mean properties of diffractive orbits, the so-called diagonal approximation, they show modifications which, in general, do not vanish in the semiclassical limit ($\hbar \rightarrow 0$). In order that this does not lead to deviations from random matrix statistics, these terms have to be cancelled by off-diagonal terms which contain information concerning correlations between

different trajectories. As remarked above, the calculation of correlations between trajectories is an unsolved problem in general systems. For the diffractive orbits that describe the influence of a scatterer, however, off-diagonal terms can be calculated explicitly. This is done in the following sections. The results show that diagonal and off-diagonal terms do indeed cancel each other. Furthermore, the results can be used to investigate parametric spectral correlations, i.e. correlations between spectra of the system for different parameter values, where the parameter is the strength of the scatterer. It is shown that the parametric spectral correlations are universal for small changes of the parameter.

2. The spectral form factor

The perturbation by a point-like scatterer is represented, formally, by a delta-potential $\hat{H} = \hat{H}_0 + \lambda \delta(\mathbf{r} - \mathbf{r}_0)$, where λ and \mathbf{r}_0 are the strength and position of the scatterer, respectively. In more than one dimension such a delta-potential is, however, not well defined. For example, it leads to a divergent expression for the Green function. The problem can be regularized by the method of self-adjoint extensions, leading to a one-parameter family of Hamiltonians. A detailed monograph with references on the history and on applications of delta-like potentials is [9]. We use in the following the property that the semiclassical approximation for the density of states has the same form as in the geometrical theory of diffraction [10–13] (see also [14, 15] for applications on spectral statistics).

We consider chaotic systems whose Hamiltonian is given in terms of a scalar and a vector potential. Billiard systems can be included in this description by letting the scalar potential be infinite outside the billiard region. The statistical distribution of the energy levels is investigated by semiclassically approximating the spectral form factor. We restrict to two-dimensional systems in order to keep the notation simple, but analogous calculations can be performed in higher dimensions.

The spectral form factor is defined as Fourier transform of the spectral two-point correlation function

$$K(\tau) = \int_{-\infty}^{\infty} \frac{d\eta}{\bar{d}(E)} \left\langle d_{\text{osc}}\left(E + \frac{\eta}{2}\right) d_{\text{osc}}\left(E - \frac{\eta}{2}\right) \right\rangle_E \exp(2\pi i \eta \tau \bar{d}(E)). \quad (1)$$

The function $d_{\text{osc}}(E) = d(E) - \bar{d}(E)$ is the oscillatory part of the density of states, and $\bar{d}(E)$ is the smooth part which is given in two dimensions by $\bar{d}(E) \sim \Sigma(E)(2\pi\hbar)^{-2}$, $E \rightarrow \infty$, where $\Sigma(E)$ is the volume of the surface of constant energy in phase space. The statistics is evaluated by averaging over an energy interval that is small in comparison to E but contains a large number of energy levels.

The semiclassical approximation for $K(\tau)$ is obtained by inserting into (1) the approximation for the oscillatory part of the density of states

$$d_{\text{osc}}(E) \approx \frac{1}{\pi\hbar} \text{Re} \sum_{\gamma} A_{\gamma} \exp\left(\frac{i}{\hbar} S_{\gamma}(E)\right). \quad (2)$$

In systems with a delta-like potential the sum in (2) runs over all periodic orbits [5], and further over all diffractive orbits that start from the scatterer and return to it an arbitrary number of times n [10–13]. An example of a double-diffractive orbit ($n = 2$) in a billiard system is shown in figure 1. For n -fold diffractive orbits the amplitude A_{γ} has an \hbar dependence of $\hbar^{n/2}$, and S_{γ} denotes the action of an orbit.

With (2) one obtains the following approximation for the spectral form factor:

$$K(\tau) = \frac{1}{2\pi\hbar\bar{d}(E)} \left\langle \sum_{\gamma, \gamma'} A_{\gamma} A_{\gamma'}^* \exp\left\{\frac{i}{\hbar} (S_{\gamma}(E) - S_{\gamma'}(E))\right\} \delta\left(T - \frac{T_{\gamma} + T_{\gamma'}}{2}\right) \right\rangle_E \quad (3)$$

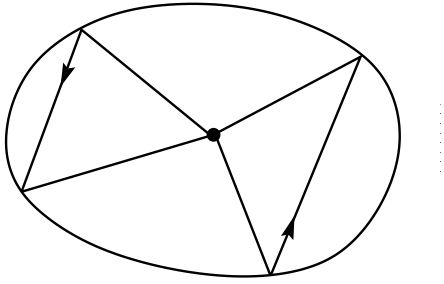


Figure 1. Example of a double-diffractive orbit.

where $T = 2\pi\hbar\bar{d}(E)\tau$, and T_γ is the period of an orbit. For small values of τ one can evaluate the double sum in (3) in the diagonal approximation [2]. In this way one obtains from the periodic orbits the correct random matrix result $K(\tau) \sim \frac{2}{\beta}\tau$, $\tau \rightarrow 0$, where $\beta = 1$ or 2 for systems with or without time-reversal symmetry, respectively.

The diagonal contributions from diffractive orbits to the form factor have been calculated in [8]. The result for n -fold diffractive orbits is

$$K_d^{(n)}(\tau) = \frac{|\mathcal{D}|^{2n} \tau^{n+1}}{(2\beta)^n n} \quad (4)$$

where \mathcal{D} is the diffraction coefficient for the diffraction on the singularity of the potential [9, 16]. It can be parametrized in the following form:

$$\mathcal{D} = \frac{2\pi}{i\pi/2 - \gamma - \log(ka/2)}. \quad (5)$$

Here $k = \sqrt{2m(E - V(r_0))/\hbar}$, r_0 is the position of the scatterer, a is a parameter describing the strength of the potential and γ is Euler's constant. In order for the terms (4) to not lead to a deviation from random matrix statistics they have to be cancelled by off-diagonal terms involving diffractive orbits. By calculating off-diagonal terms for order τ^2 and τ^3 we show in the following that such a cancellation does indeed occur.

We note that the diffraction coefficient satisfies the identities

$$|\mathcal{D}|^2 = -4 \operatorname{Im} \mathcal{D} \quad |\mathcal{D}|^4 = 8(|\mathcal{D}|^2 - \operatorname{Re} \mathcal{D}^2) \quad (6)$$

that will be used in the following. The first of these relations expresses the conservation of probability and the second is a consequence of the first one.

3. First-order correction

The first-order correction to the diagonal approximation for the form factor arises from off-diagonal terms in (3) between periodic orbits and single-diffractive orbits. In leading order, these orbits are only correlated if the diffractive orbit follows the periodic orbit very closely. This happens, if the diffractive orbit is almost periodic, i.e. if the final momentum is almost identical to the initial momentum. An example is shown in figure 2.

The periodic orbit can be described by linearizing the motion around the diffractive orbit. The condition that a trajectory in the vicinity of the diffractive orbit is periodic leads to the following equation:

$$\begin{pmatrix} \delta \\ p_v \gamma \end{pmatrix} = M \begin{pmatrix} \delta \\ p_v(\gamma - \varepsilon) \end{pmatrix}. \quad (7)$$

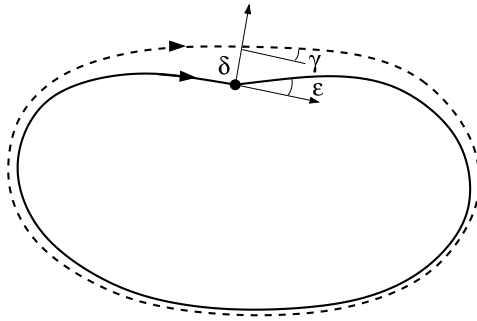


Figure 2. A diffractive orbit (full curve) that is almost periodic and a nearby periodic orbit (broken curve). The local coordinate system is oriented along the final direction of the diffractive orbit.

Here M is the stability matrix of the diffractive orbit, ε is the angle between the initial and final direction of the diffractive orbit, γ is the angle between the direction along the periodic orbit and the final direction of the diffractive orbit, and δ is the spacial distance between periodic and diffractive orbit (see figure 2). The quantity p_v is defined as mass times velocity, $p_v = mv$. The index v is used in order to distinguish it from the canonical momentum p in systems with a magnetic field. The stability matrix for the motion in a magnetic field is discussed in the appendix.

In the linear approximation the difference in actions is obtained by expanding the action up to second order

$$\Delta S(E) = S^{\text{po}}(E) - S^{\text{ldo}}(E) \approx -p_v \delta \varepsilon + \frac{1}{2}(\Delta p_f - \Delta p_i) \delta = -\frac{1}{2} \delta \varepsilon p_v \quad (8)$$

where Δp_f and Δp_i are the differences between the initial and final momenta of the periodic orbit and the diffractive orbit, respectively. The solution of the linear equation (7) yields the following relation between δ and ε :

$$\delta = \frac{M_{12}}{\text{Tr } M - 2} \varepsilon p_v \quad (9)$$

so that $\Delta S(E)$ depends quadratically on ε .

With this approximation the off-diagonal terms are calculated. The amplitude of the diffractive orbit is given by

$$A_{\gamma}^{\text{ldo}} = \frac{T_{\gamma} \mathcal{D}}{4\pi p_v} \sqrt{\frac{2\pi \hbar}{|(M_{\gamma})_{12}|}} \exp \left\{ -i \frac{\pi}{2} v_{\gamma} - i \frac{3\pi}{4} \right\} \quad (10)$$

where T_{γ} is the time along the orbit, M_{γ} is its stability matrix, and v_{γ} is the number of conjugate points along the orbit. For the periodic orbit the corresponding amplitude is

$$A_{\gamma}^{\text{po}} = \frac{T_{\gamma}}{\sqrt{|\text{Tr } M_{\gamma} - 2|}} \exp \left\{ -i \frac{1}{2} \pi \mu_{\gamma} \right\}. \quad (11)$$

The stability matrix is the same for both orbits in leading order, but the Maslov index of the periodic orbit can differ from the number of conjugate points v_{γ} by 1 [5]

$$\mu_{\gamma} = v_{\gamma} + \frac{1}{2}(1 - \kappa_{\gamma}) \quad \kappa_{\gamma} = \text{sign} \left(\frac{(M_{\gamma})_{12}}{\text{Tr } M_{\gamma} - 2} \right). \quad (12)$$

In the following we sum over all diffractive orbits that are almost periodic. This is done by first applying the sum rule for diffractive orbits for which the angle difference between the

initial and the final direction has a fixed value ε , and then integrating over the angle ε . The sum rule is given by [8]

$$\sum_{\gamma}^{(\varepsilon)} \frac{1}{|(M_{\gamma})_{12}|} \delta(T - T_{\gamma}) \approx \frac{2\pi p_v^2}{\Sigma(E)} \quad (13)$$

where $\Sigma(E)$ is the volume of the energy shell. (It is implied in (13) that the left-hand side is smoothed over small intervals of T and ε in order to obtain a non-singular expression.)

Finally, one has to determine the multiplicity factor of the contribution. First, each off-diagonal term in (3) has a corresponding complex conjugate partner. If the summation is carried out over only one of these terms one has to take twice the real part of the sum. Furthermore, the periodic orbit and the diffractive orbit both have multiplicities $2\beta^{-1}$, but a particular constellation occurs $2\beta^{-1}$ times in the sum over ε (for systems with time-reversal symmetry for ε and $-\varepsilon$), so the total multiplicity is $g = 2\beta^{-1}$.

Inserting the amplitudes (10) and (11), and the action difference (8) with (9) into (3) we obtain

$$\begin{aligned} K_{\text{off}}^{(1)}(\tau) &= \frac{g}{2\pi\hbar\bar{d}(E)} 2 \operatorname{Re} \int_{-\infty}^{\infty} d\varepsilon \sum_{\gamma}^{(\varepsilon)} A_{\gamma}^{\text{ldo}} (A_{\gamma}^{\text{pdo}})^* \exp\left(-\frac{i}{\hbar} \Delta S_{\gamma}(E)\right) \delta(T - T_{\gamma}) \\ &= \frac{4}{2\pi\hbar\bar{d}(E)\beta} \operatorname{Re} \int_{-\infty}^{\infty} d\varepsilon \sum_{\gamma}^{(\varepsilon)} \frac{T_{\gamma}^2 \mathcal{D} \sqrt{2\pi\hbar} \delta(T - T_{\gamma})}{4\pi p_v \sqrt{|(M_{\gamma})_{12}| (\operatorname{Tr} M_{\gamma} - 2)}} \\ &\quad \times \exp\left\{ \frac{i\varepsilon^2 p_v^2}{2\hbar} \frac{(M_{\gamma})_{12}}{\operatorname{Tr} M_{\gamma} - 2} - i\frac{\pi}{4} (2 + \kappa_{\gamma}) \right\} \\ &= \frac{4}{2\pi\hbar\bar{d}(E)\beta} \operatorname{Re} \int_{-\infty}^{\infty} d\varepsilon' \sum_{\gamma}^{(\varepsilon')} \frac{T_{\gamma}^2 \mathcal{D} \sqrt{2\pi\hbar} \delta(T - T_{\gamma})}{4\pi p_v |(M_{\gamma})_{12}|} \exp\left[-\frac{\varepsilon'^2 p_v^2}{2\hbar} - i\frac{\pi}{2}\right] \\ &= \frac{2T^2}{2\pi\hbar\bar{d}(E)\beta} \operatorname{Im} \int_{-\infty}^{\infty} d\varepsilon' \frac{\mathcal{D} p_v \sqrt{2\pi\hbar}}{\Sigma(E)} \exp\left[-\frac{\varepsilon'^2 p_v^2}{2\hbar}\right] \\ &= \frac{2\tau^2}{\beta} \operatorname{Im} \mathcal{D}. \end{aligned} \quad (14)$$

The integration over ε can be carried out from minus to plus infinity since the main contribution comes from the vicinity of $\varepsilon = 0$. Furthermore, the following steps have been carried out. First the integration variable has been changed to make the exponent independent of the stability matrix. Then the sum rule (13) has been applied, assuming that the distribution of angles between initial and final momenta of a diffractive orbit is independent of the distribution of the elements of the stability matrix

$$\sum_{\gamma}^{(\varepsilon)} \frac{1}{|(M_{\gamma})_{12}|} \delta(T - T_{\gamma}) \approx \sum_{\gamma}^{(\varepsilon')} \frac{1}{|(M_{\gamma})_{12}|} \delta(T - T_{\gamma}) \quad \varepsilon' = \varepsilon \sqrt{\frac{-i(M_{\gamma})_{12}}{\operatorname{Tr} M_{\gamma} - 2}} \quad (15)$$

and finally the integration has been carried out.

$K_{\text{off}}^{(1)}(\tau)$ is the leading-order correction to the diagonal approximation for the form factor and it cancels exactly the diagonal contribution from single-diffractive orbits ((4) with $n = 1$). This can be seen by using (6)

$$K_{\text{d}}^{(1)}(\tau) + K_{\text{off}}^{(1)}(\tau) = \frac{|\mathcal{D}|^2}{2\beta} \tau^2 + \frac{2}{\beta} \tau^2 \operatorname{Im} \mathcal{D} = 0. \quad (16)$$

It shows that the presence of a point-like scatterer does not modify the spectral form factor up to order τ^2 in systems with a chaotic classical limit.

In order to find the geometries of orbits which contribute to a given order in τ it is helpful to count the orders of \hbar . The m th-order off-diagonal correction to the form factor is a τ^{m+1} term with a coefficient that has to be \hbar -independent. The prefactor of the double sum over orbits in (3) is of order \hbar^{-1} and the product of the amplitudes of an n_1 -fold and an n_2 -fold diffractive orbit is of order $\hbar^{-(n_1+n_2)/2}$, where periodic orbits are denoted here as 0-fold diffractive orbits. The conversion of time T^{m+1} into τ^{m+1} gives an order \hbar^{m+1} , which yields altogether an order of $\hbar^{(2m-n_1-n_2)/2}$. Furthermore, every integration over a small parameter ε gives an additional order $\hbar^{-1/2}$, if the action difference is quadratic in this parameter. Consequently, $2m - n_1 - n_2$ small parameters are necessary in order for the prefactor of $\tau^{(m+1)}$ to be \hbar -independent. For the first-order correction in this section ($m = n_1 = 1$ and $n_2 = 0$) this estimate gives one small parameter ε .

4. Second-order corrections

For the second-order corrections we consider orbits that return twice to the region in coordinate space from which they started. These orbits are close to double-diffractive orbits. Double-diffractive orbits have the semiclassical amplitude

$$A_\gamma^{2\text{do}} = \frac{\hbar T_\gamma \mathcal{D}^2}{16\pi p_v^2} \frac{1}{\sqrt{|(R_\gamma)_{12}(L_\gamma)_{12}|}} \exp \left\{ -i \frac{\pi}{2} (\nu_{\gamma,L} + \nu_{\gamma,R}) - i \frac{3\pi}{2} \right\} \quad (17)$$

where T_γ is the total time along the trajectory, L_γ and R_γ are the stability matrices for the two loops ('left' and 'right'), and $\nu_{\gamma,L}$ and $\nu_{\gamma,R}$ are the number of conjugate points along the loops. Equation (17) is the amplitude for one particular sequence in which the loops are traversed. In systems without time-reversal symmetry the degeneracy of the trajectory is thus two, meaning that there is another trajectory with exactly the same semiclassical amplitude and action. This trajectory traverses first the second loop of γ , and then the first loop of γ . In systems with time-reversal symmetry the degeneracy is eight.

The sum rule for double-diffractive orbits is given by [8]

$$\sum_{\gamma}^{(\varepsilon_1, \dots, \varepsilon_n)} \frac{1}{|(R_\gamma)_{12}(L_\gamma)_{12}|} \delta(T - T_\gamma) \approx \frac{(2\pi p_v)^4}{\Sigma(E)^2} \frac{T}{(2\pi)^n} \quad (18)$$

if there are n restrictions to the four directions of the velocities at the point from which the trajectories start and to which they return. As will be seen in the following, it follows from this sum rule that the contributions are of order τ^3 (there is a factor T from every semiclassical amplitude, and a factor T from the sum rule).

There are several possibilities in which a double-diffractive orbit can have an action which is almost identical to the action of a single-diffractive or a periodic orbit. A necessary condition is that there is always at least one small relative angle between the different initial and final directions of the orbit at the scattering point. In order to find the relevant cases one has to consider all possibilities and take into account the \hbar -argument that was given at the end of section 3. The result is that there are three relevant configurations for systems without time-reversal symmetry and five configurations for systems with time-reversal symmetry. They are discussed in the next sections.

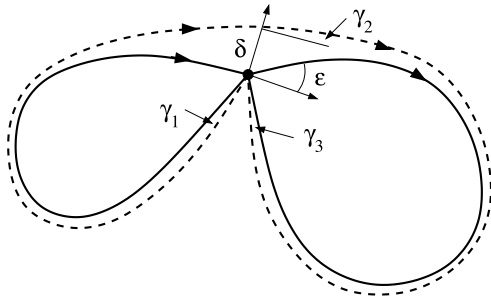


Figure 3. A double-diffractive orbit (full curve) for which one initial direction deviates by a small angle ε from one final direction, and a nearby single-diffractive orbit (broken curve).

4.1. Correlations between double-diffractive and single-diffractive orbits

Correlations between double- and single-diffractive orbits exist if the double-diffractive orbit is almost single-diffractive. This occurs if the final velocity of one loop deviates by a small angle ε from the initial velocity of the other loop. An example is shown in figure 3. There is only one small parameter here which agrees with the estimate $2m - n_1 - n_2$ for $m = 2$, $n_1 = 2$ and $n_2 = 1$.

Further calculations are done analogously to the previous section. The motion in the vicinity of the double-diffractive orbit is linearized and one obtains in this approximation a condition for the neighbouring single-diffractive orbit

$$\begin{pmatrix} \delta \\ p_v \gamma_2 \end{pmatrix} = L \begin{pmatrix} 0 \\ p_v \gamma_1 \end{pmatrix} \quad \begin{pmatrix} 0 \\ p_v \gamma_3 \end{pmatrix} = R \begin{pmatrix} \delta \\ p_v (\gamma_2 - \varepsilon) \end{pmatrix}. \quad (19)$$

The angles γ_1 , γ_2 , γ_3 and the distance δ are shown in figure 3. The difference in action is obtained by expanding the action up to second order

$$\Delta S(E) = S^{1\text{do}}(E) - S^{2\text{do}}(E) = -\frac{1}{2} \delta \varepsilon p_v = -\frac{\varepsilon^2 p_v^2}{2} \frac{L_{12} R_{12}}{M_{12}} \quad (20)$$

where $M = RL$ is the stability matrix of the single-diffractive orbit. The last step in (20) follows from the solution of (19) for δ .

The number of conjugate points ν_γ along the single-diffractive orbit can differ from the sum of the number of conjugate points along the two loops, $\nu_{\gamma,L}$ and $\nu_{\gamma,R}$, by 1. The general condition for this is

$$\nu_\gamma = \nu_{\gamma,L} + \nu_{\gamma,R} + \frac{1}{2}(1 - \sigma_\gamma) \quad \sigma_\gamma = \text{sign} \left(\frac{(L_\gamma)_{12}(R_\gamma)_{12}}{(M_\gamma)_{12}} \right). \quad (21)$$

The single- and double-diffractive orbits have multiplicity $2\beta^{-1}$ and $8\beta^{-2}$, respectively, but each configuration occurs for $2\beta^{-1}$ different values of ε . Therefore, the total multiplicity is $g = 8\beta^{-2}$. The contribution to the form factor from all pairs of orbits is thus given by

$$\begin{aligned} K_{\text{off}}^{(2a)}(\tau) &= \frac{g}{2\pi\hbar\bar{d}(E)} 2 \text{Re} \int_{-\infty}^{\infty} d\varepsilon \sum_{\gamma}^{(\varepsilon)} A_\gamma^{2\text{do}} (A_\gamma^{1\text{do}})^* \exp \left(-\frac{i}{\hbar} \Delta S_\gamma(E) \right) \delta(T - T_\gamma) \\ &= \frac{16}{2\pi\hbar\bar{d}(E)\beta^2} \text{Re} \int_{-\infty}^{\infty} d\varepsilon \sum_{\gamma}^{(\varepsilon)} \frac{\hbar T_\gamma^2 \mathcal{D}^2 \mathcal{D}^* \sqrt{2\pi\hbar} \delta(T - T_\gamma)}{64\pi^2 p_v^3 \sqrt{|(L_\gamma)_{12}(R_\gamma)_{12}(M_\gamma)_{12}|}} \\ &\quad \times \exp \left\{ \frac{i\varepsilon^2 p_v^2}{2\hbar} \frac{(L_\gamma)_{12}(R_\gamma)_{12}}{(M_\gamma)_{12}} - i\frac{\pi}{4}(2 + \sigma_\gamma) \right\} \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{2\pi\hbar\bar{d}(E)\beta^2} \operatorname{Re} \sum_{\gamma}^{(\varepsilon')} \frac{\hbar^2 T^2 \mathcal{D}^2 \mathcal{D}^* \delta(T - T_{\gamma})}{2\pi p_v^4 |(L_{\gamma})_{12} (R_{\gamma})_{12}|} e^{-i\frac{\pi}{2}} \\
&= \frac{\tau^3}{\beta^2} \operatorname{Im}(\mathcal{D}^2 \mathcal{D}^*). \tag{22}
\end{aligned}$$

Here we have slightly abbreviated the procedure of (14) and performed the integration directly.

4.2. Correlations between double-diffractive orbits and periodic orbits

In order for a double-diffractive orbit to be close to a periodic orbit it has to be almost periodic. This means that the final direction of each loop has to be almost identical to the initial direction of the other loop, so there are two small relative angles as shown in figure 4.

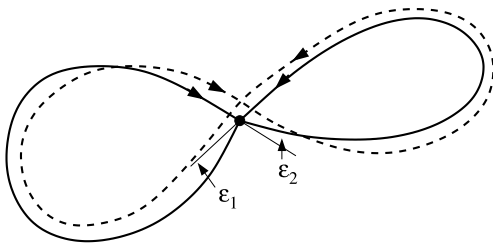


Figure 4. A double-diffractive orbit (full curve) for which the two initial directions deviate by small angles ε_1 and ε_2 from the two final directions, and a nearby periodic orbit (broken curve).

The linearization of the motion in the vicinity of the double-diffractive orbit leads to the following condition for the periodic orbit:

$$\begin{pmatrix} \delta_2 \\ p_v \gamma_2 \end{pmatrix} = L \begin{pmatrix} \delta_1 \\ p_v (\gamma_1 - \varepsilon_1) \end{pmatrix} \quad \begin{pmatrix} \delta_1 \\ p_v \gamma_1 \end{pmatrix} = R \begin{pmatrix} \delta_2 \\ p_v (\gamma_2 - \varepsilon_2) \end{pmatrix}. \tag{23}$$

The angles γ_i and distances δ_i are defined analogously as before in terms of the local coordinate systems that are oriented along the two final directions of the diffractive orbit.

The difference in actions is given by

$$\begin{aligned}
\Delta S(E) &= S^{\text{po}}(E) - S^{2\text{do}}(E) \\
&= -\frac{1}{2} p_v (\delta_1 \varepsilon_1 + \delta_2 \varepsilon_2) \\
&= -\frac{1}{2} p_v^2 \frac{(RL)_{12} \varepsilon_1^2 + (LR)_{12} \varepsilon_2^2 + 2(L_{12} + R_{12}) \varepsilon_1 \varepsilon_2}{\operatorname{Tr} M - 2} \tag{24}
\end{aligned}$$

where $M = RL$ is the stability matrix for the periodic orbit. Equation (24) can be written in terms of a symmetric matrix A such that $\Delta S(E) = -\frac{1}{2} p_v^2 \sum_{i,j} A_{ij} \varepsilon_i \varepsilon_j$. The number of negative eigenvalues of A is given by the number of sign changes in the sequence of sub-determinants $(1, A_{11}, \det A)$. With $A_{11} = M_{12}/(\operatorname{Tr} M - 2)$ and $\det A = L_{12} R_{12}/(\operatorname{Tr} M - 2)$ one finds that the two signs of the eigenvalues are given by

$$\kappa = \operatorname{sign} \left(\frac{M_{12}}{\operatorname{Tr} M - 2} \right) \quad \sigma = \operatorname{sign} \left(\frac{L_{12} R_{12}}{M_{12}} \right). \tag{25}$$

The Maslov index of the periodic orbit can now differ by zero, one or two from the sum of conjugate points along the left and right loop. The criterion for this is the combination of (12) and (21) and has the form

$$\mu = \nu_L + \nu_R + \frac{1}{2}(2 - \kappa - \sigma) \tag{26}$$

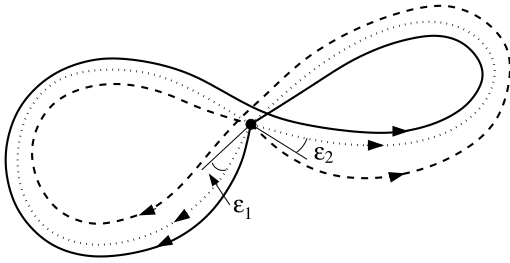


Figure 5. A double-diffractive orbit (dotted curve) for which the two initial directions deviate by small angles ε_1 and ε_2 from the two final directions, and two nearby single-diffractive orbits (full and broken curves).

where κ and σ are given in (25).

The multiplicities of double-diffractive and periodic orbits are $8\beta^{-2}$ and $2\beta^{-1}$, respectively, but a particular configuration of them occurs $4\beta^{-1}$ times in the integral over the angles (for example, for systems without time-reversal symmetry the two angles can be interchanged), so the total multiplicity is $g = 4\beta^{-2}$.

After inserting the amplitudes (17) and (11) and the action difference (24) into (3) we obtain

$$\begin{aligned}
 K_{\text{off}}^{(2b)}(\tau) &= \frac{g}{2\pi\hbar\bar{d}(E)} 2 \operatorname{Re} \int_{-\infty}^{\infty} d\varepsilon_1 d\varepsilon_2 \sum_{\gamma}^{(\varepsilon_1, \varepsilon_2)} A_{\gamma}^{2\text{do}} (A_{\gamma}^{\text{po}})^* \exp\left(-\frac{i}{\hbar} \Delta S_{\gamma}(E)\right) \delta(T - T_{\gamma}) \\
 &= \frac{8}{2\pi\hbar\bar{d}(E)\beta^2} \operatorname{Re} \int_{-\infty}^{\infty} d\varepsilon_1 d\varepsilon_2 \sum_{\gamma}^{(\varepsilon_1, \varepsilon_2)} \frac{\hbar T_{\gamma}^2 \mathcal{D}^2 \delta(T - T_{\gamma}) e^{\frac{i\pi}{2\hbar} \sum A_{i,j} \varepsilon_i \varepsilon_j - i\frac{\pi}{4}(4+\sigma_{\gamma}+\kappa_{\gamma})}}{16\pi p_v^2 \sqrt{|(L_{\gamma})_{12}(R_{\gamma})_{12}(\operatorname{Tr} M_{\gamma} - 2)|}} \\
 &= \frac{1}{2\pi\hbar\bar{d}(E)\beta^2} \operatorname{Re} \sum_{\gamma}^{(\varepsilon'_1, \varepsilon'_2)} \frac{\hbar^2 T^2 \mathcal{D}^2 \delta(T - T_{\gamma})}{p_v^4 |(L_{\gamma})_{12}(R_{\gamma})_{12}|} e^{-i\pi} \\
 &= -\frac{\tau^3}{\beta^2} \operatorname{Re}(\mathcal{D}^2). \tag{27}
 \end{aligned}$$

4.3. Correlations between pairs of single-diffractive orbits

For exactly the same kind of double-diffractive orbits as in the last section, there is one further type of correlation that has to be considered. It occurs because there are two possible ways in which the double-diffractive orbit can be deformed into a nearby single-diffractive orbit, and consequently, there are correlations between these single-diffractive orbits.

The action difference between the two orbits can be obtained from the action difference between each of these orbits and the double-diffractive orbit (20)

$$\Delta S(E) = S^{\text{ldo},1}(E) - S^{\text{ldo},2}(E) = -\frac{1}{2}\delta_1 \varepsilon_1 p_v + \frac{1}{2}\delta_2 \varepsilon_2 p_v = -\frac{1}{2}p_v^2 \left(\frac{L_{12}R_{12}}{N_{12}} \varepsilon_1^2 - \frac{L_{12}R_{12}}{M_{12}} \varepsilon_2^2 \right) \tag{28}$$

where $M = RL$ and $N = LR$ are the stability matrices of the two single-diffractive orbits. The number of conjugate points along the orbits are given by

$$\begin{aligned}
 \nu_1 &= \nu_L + \nu_R + \frac{1}{2}(1 - \sigma_1) & \nu_2 &= \nu_L + \nu_R + \frac{1}{2}(1 - \sigma_2) \\
 \sigma_1 &= \operatorname{sign}\left(\frac{L_{12}R_{12}}{N_{12}}\right), & \sigma_2 &= \operatorname{sign}\left(\frac{L_{12}R_{12}}{M_{12}}\right). \tag{29}
 \end{aligned}$$

The multiplicities of the two orbits is both $2\beta^{-1}$ and, as before, the configuration occurs $4\beta^{-1}$ times in the double integral over the angles, so the total multiplicity is $g = \beta^{-1}$.

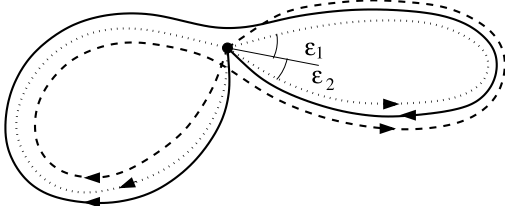


Figure 6. A double-diffractive orbit (dotted curve) with one loop for which the initial direction and the reversed final direction deviate by small angles ε_1 and ε_2 from the final direction of the other loop. Furthermore, two nearby single-diffractive orbits (full and broken curves), one of which traverses the second loop in the opposite direction (full curve).

Inserting the amplitude (10) with stability matrix M and N , respectively, and the action difference (28) into (3) one obtains

$$\begin{aligned}
 K_{\text{off}}^{(2c)}(\tau) &= \frac{g}{2\pi\hbar\bar{d}(E)} 2 \operatorname{Re} \int_{-\infty}^{\infty} d\varepsilon_1 d\varepsilon_2 \sum_{\gamma}^{(\varepsilon_1, \varepsilon_2)} A_{\gamma}^{\text{ldo}, 2} (A_{\gamma}^{\text{ldo}, 1})^* \exp\left(-\frac{i}{\hbar} \Delta S_{\gamma}(E)\right) \delta(T - T_{\gamma}) \\
 &= \frac{2}{2\pi\hbar\bar{d}(E)\beta} \operatorname{Re} \int_{-\infty}^{\infty} d\varepsilon_1 d\varepsilon_2 \\
 &\quad \times \sum_{\gamma}^{(\varepsilon_1, \varepsilon_2)} \frac{2\pi\hbar T_{\gamma}^2 |\mathcal{D}|^2 \delta(T - T_{\gamma}) e^{-\frac{i}{\hbar} \Delta S_{\gamma}(E) - i\frac{\pi}{4}(\sigma_{\gamma, 1} - \sigma_{\gamma, 2})}}{16\pi^2 p_v^2 \sqrt{|(M_{\gamma})_{12}(N_{\gamma})_{12}|}} \\
 &= \frac{1}{2\pi\hbar\bar{d}(E)\beta} \operatorname{Re} \sum_{\gamma}^{(\varepsilon'_1, \varepsilon'_2)} \frac{\hbar^2 T^2 |\mathcal{D}|^2 \delta(T - T_{\gamma})}{2p_v^4 |(L_{\gamma})_{12}(R_{\gamma})_{12}|} \\
 &= \frac{\tau^3}{2\beta} |\mathcal{D}|^2. \tag{30}
 \end{aligned}$$

The contributions $K_{\text{off}}^{(2a)}(\tau)$, $K_{\text{off}}^{(2b)}(\tau)$ and $K_{\text{off}}^{(2c)}(\tau)$ are the only second-order off-diagonal corrections in systems without time-reversal symmetry. As will be shown in the following, these contributions cancel exactly the diagonal term $K_{\text{d}}^{(2)}(\tau)$ for $\beta = 2$. For systems with time-reversal symmetry there are further contributions. They arise from the possibility that one trajectory can follow one loop of another trajectory in the same direction, but the other loop in the time-reversed direction. We assume in the following that the relevant contributions come from orbits which are close to double-diffractive orbits in coordinate space and we evaluate their contributions in the next two subsections.

4.4. Correlations between pairs of single-diffractive orbits involving time-reversed loops

The first possibility involves two single-diffractive orbits. These orbits follow closely a double-diffractive orbit and traverse one loop in the same direction and the other loop in the opposite direction. In order that this can occur there must be one loop which has a very small opening angle, and it must be almost aligned to the final direction of the other loop as shown in figure 6.

The action difference and the indices are given by equations (28) and (29) but now with $M = RL$ and $N = R^i L$, where R^i is the stability matrix for the time-reversed second loop. In term of the elements of R the matrix R^i is given by

$$R^i = \begin{pmatrix} R_{22} & R_{12} \\ R_{21} & R_{11} \end{pmatrix}. \tag{31}$$

The multiplicity of each orbit is two, and the configuration occurs two times in the integral over the angles, so the total multiplicity is $g = 2$. Inserting the amplitude (10) with stability

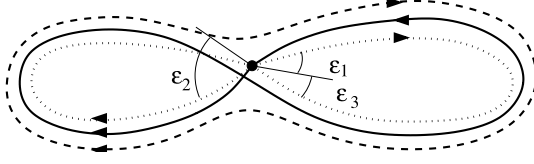


Figure 7. A double-diffractive orbit (dotted curve) for which all initial and final directions lie almost in one line. Furthermore, a nearby periodic orbit (broken curve), and a nearby single-diffractive orbit (full curve) which traverses the second loop in the opposite direction.

matrix M and N , respectively, and the action difference (28) into (3) results in

$$\begin{aligned}
 K_{\text{off}}^{(2d)}(\tau) &= \frac{g}{2\pi\hbar\bar{d}(E)} 2 \operatorname{Re} \int_{-\infty}^{\infty} d\varepsilon_1 d\varepsilon_2 \sum_{\gamma}^{(\varepsilon_1, \varepsilon_2)} A_{\gamma}^{\text{ldo}, 2} (A_{\gamma}^{\text{ldo}, 1})^* \\
 &\quad \times \exp\left(-\frac{i}{\hbar} \Delta S_{\gamma}(E)\right) \delta(T - T_{\gamma}) \\
 &= \frac{4}{2\pi\hbar\bar{d}(E)} \operatorname{Re} \int_{-\infty}^{\infty} d\varepsilon_1 d\varepsilon_2 \\
 &\quad \times \sum_{\gamma}^{(\varepsilon_1, \varepsilon_2)} \frac{2\pi\hbar T_{\gamma}^2 |\mathcal{D}|^2 \delta(T - T_{\gamma}) e^{-\frac{i}{\hbar} \Delta S_{\gamma}(E) - i\frac{\pi}{4}(\sigma_{\gamma, 1} - \sigma_{\gamma, 2})}}{16\pi^2 p_v^2 \sqrt{|(M_{\gamma})_{12}(N_{\gamma})_{12}|}} \\
 &= \frac{1}{2\pi\hbar\bar{d}(E)} \operatorname{Re} \sum_{\gamma}^{(\varepsilon'_1, \varepsilon'_2)} \frac{\hbar^2 T^2 |\mathcal{D}|^2 \delta(T - T_{\gamma})}{p_v^4 |(L_{\gamma})_{12}(R_{\gamma})_{12}|} \\
 &= \tau^3 |\mathcal{D}|^2.
 \end{aligned} \tag{32}$$

4.5. Correlations between single-diffractive orbits and periodic orbits involving time-reversed loops

The last relevant configuration occurs if all initial and final velocities of the double-diffractive orbit lie almost in one line as in figure 7. Then there exist neighbouring single-diffractive and periodic orbits which follow one loop in the same direction and the other loop in the opposite direction.

The action difference between the periodic orbit and the diffractive orbit is obtained from (20) and (24),

$$\begin{aligned}
 \Delta S(E) &= S^{\text{po}}(E) - S^{\text{ldo}}(E) \\
 &= -\frac{p_v}{2} (\delta_1 \varepsilon_1 + \delta_2 \varepsilon_2) + \frac{p_v}{2} \delta_3 \varepsilon_3 \\
 &= -\frac{p_v^2}{2} \frac{(RL)_{12} \varepsilon_1^2 + (LR)_{12} \varepsilon_2^2 + 2(L_{12} + R_{12}) \varepsilon_1 \varepsilon_2}{\operatorname{Tr} M - 2} + \frac{1}{2} \varepsilon_3^2 p_v^2 \frac{L_{12} R_{12}}{N_{12}}
 \end{aligned} \tag{33}$$

where $M = RL$ and $N = R^i L$ are the stability matrices of the periodic and diffractive orbit, respectively, and the indices ν and μ of the diffractive and the periodic orbit are

$$\begin{aligned}
 \nu &= \nu_L + \nu_R + \frac{1}{2}(1 - \sigma_2) & \mu &= \nu_L + \nu_R + \frac{1}{2}(2 - \kappa - \sigma_1) \\
 \sigma_2 &= \operatorname{sign}\left(\frac{L_{12} R_{12}}{N_{12}}\right) & \kappa &= \operatorname{sign}\left(\frac{M_{12}}{\operatorname{Tr} M - 2}\right) & \sigma_1 &= \operatorname{sign}\left(\frac{L_{12} R_{12}}{M_{12}}\right).
 \end{aligned} \tag{34}$$

For each double-diffractive orbit like that in figure 7 there are two periodic orbits of multiplicity two and four single-diffractive orbits of multiplicity two in the vicinity, but only half of the possible pairs involve exactly one time-reversed loop which makes a total of 16.

Furthermore, the double-diffractive orbit occurs eight times in the integral over the angles and the total multiplicity is thus $g = 2$.

Inserting the amplitudes (10) and (11) and the action difference (33) into (3) yields

$$\begin{aligned}
K_{\text{off}}^{(2e)}(\tau) &= \frac{g}{2\pi\hbar\bar{d}(E)} 2 \operatorname{Re} \int_{-\infty}^{\infty} d\varepsilon_1 d\varepsilon_2 d\varepsilon_3 \sum_{\gamma}^{(\varepsilon_1, \varepsilon_2, \varepsilon_3)} A_{\gamma}^{\text{ldo}} (A_{\gamma}^{\text{po}})^* \\
&\quad \times \exp\left(-\frac{i}{\hbar} \Delta S_{\gamma}(E)\right) \delta(T - T_{\gamma}) \\
&= \frac{4}{2\pi\hbar\bar{d}(E)} \operatorname{Re} \int_{-\infty}^{\infty} d\varepsilon_1 d\varepsilon_2 d\varepsilon_3 \\
&\quad \times \sum_{\gamma}^{(\varepsilon_1, \varepsilon_2, \varepsilon_3)} \frac{T_{\gamma}^2 \mathcal{D} \sqrt{2\pi\hbar} \delta(T - T_{\gamma}) e^{-\frac{i}{\hbar} \Delta S_{\gamma}(E) - i\frac{\pi}{4}(2 - \sigma_{\gamma, 2} + \kappa_{\gamma} + \sigma_{\gamma, 1})}}{4\pi p_v \sqrt{|(M_{\gamma})_{12}(\operatorname{Tr} M_{\gamma} - 2)|}} \\
&= \frac{1}{2\pi\hbar\bar{d}(E)} \operatorname{Re} \sum_{\gamma}^{(\varepsilon'_1, \varepsilon'_2, \varepsilon'_3)} \frac{4\pi\hbar^2 T^2 \mathcal{D} \delta(T - T_{\gamma})}{p_v^4 |(L_{\gamma})_{12}(R_{\gamma})_{12}|} e^{-i\frac{\pi}{2}} \\
&= 2\tau^3 \operatorname{Im} \mathcal{D}. \tag{35}
\end{aligned}$$

The sum of all contributions can be written in the form

$$\begin{aligned}
K^{(2)}(\tau) &= K_{\text{d}}^{(2)}(\tau) + K_{\text{off}}^{(2a)}(\tau) + K_{\text{off}}^{(2b)}(\tau) + K_{\text{off}}^{(2c)}(\tau) + K_{\text{off}}^{(2d)}(\tau) + K_{\text{off}}^{(2e)}(\tau) \\
&= \frac{\tau^3}{\beta^2} \left(\frac{1}{8} |\mathcal{D}|^4 + |\mathcal{D}|^2 \operatorname{Im} \mathcal{D} - \operatorname{Re} \mathcal{D}^2 + \frac{\beta}{2} |\mathcal{D}|^2 + (2 - \beta) |\mathcal{D}|^2 + 2(2 - \beta) \operatorname{Im} \mathcal{D} \right) \\
&= \frac{\tau^3}{\beta^2} \left(\frac{1}{8} |\mathcal{D}|^4 + |\mathcal{D}|^2 \operatorname{Im} \mathcal{D} - \operatorname{Re} \mathcal{D}^2 + |\mathcal{D}|^2 \right) \\
&= 0 \tag{36}
\end{aligned}$$

which can be seen by using (6). This shows that off-diagonal terms also cancel the diagonal term in this order. It implies that the form factor is determined by periodic orbits alone, because the different terms which involve diffractive orbits cancel each other. This might be true also for other point-like sources of diffraction such as, for example, Aharonov–Bohm flux lines (see [17]), although a quantitative analysis would require the use of uniform approximations here.

5. Universality in parametric correlations

The cancellation of off-diagonal and diagonal terms conforms with the expected universality of spectral statistics in chaotic systems. Universality is, however, not only expected in the properties of single systems, but also in the way in which system properties vary when a parameter of the system is changed. For example, random matrix theory makes predictions about correlations between densities of states for different parameter values. The semiclassical calculation of diagonal and off-diagonal terms allows us to test this prediction for systems with a point-like scatterer where the parameter is the strength of the scatterer.

In analogy to the spectral form factor, a parametric form factor can be defined as the Fourier transform of the parametric two-point correlation function,

$$K(\tau, x) = \int_{-\infty}^{\infty} \frac{d\eta}{\bar{d}(E)} \left\langle d_{\text{osc}} \left(E + \frac{\eta}{2}, X + \frac{x}{2} \right) d_{\text{osc}} \left(E - \frac{\eta}{2}, X - \frac{x}{2} \right) \right\rangle_E \exp(2\pi i \eta \tau \bar{d}(E)). \tag{37}$$

Here x is the parameter difference between two systems. In order for this statistics to be universal the parameter has to be chosen in a particular way. The requirement is that the variance of the velocities, the derivatives of the unfolded energies with respect to the parameter ($\partial \varepsilon_n / \partial x$), is equal to unity. The unfolded energies are obtained from the quantum energies of the system by a scaling that leads to mean level distance of one.

For random matrix ensembles, the parametric two-point correlation function was derived in [18] in the context of disordered metallic systems. For the GUE result, which we discuss first, the Fourier transform in (37) can be evaluated in closed form. It results in

$$K^{\text{GUE}}(\tau, x) = \begin{cases} \frac{\sinh(2\pi^2 x^2 \tau^2)}{2\pi^2 x^2 \tau} \exp(-2\pi^2 x^2 \tau) & \text{if } \tau < 1 \\ \frac{\sinh(2\pi^2 x^2 \tau)}{2\pi^2 x^2 \tau} \exp(-2\pi^2 x^2 \tau^2) & \text{if } \tau > 1 \end{cases} \quad (38)$$

and has for small values of τ the expansion

$$K^{\text{GUE}}(\tau, x) = \tau - 2\pi^2 x^2 \tau^2 + 2\pi^4 x^4 \tau^3 + \dots \quad (39)$$

We examine in the following whether the perturbation by a point-like scatterer leads to universal correlations. For large parameter differences x the parametric correlations for these systems cannot be expected to be universal. The treatment of a delta-scatterer by the method of self-adjoint extensions leads to a quantization condition with the property that there is exactly one eigenvalue of the perturbed system within each pair of neighbouring eigenvalues of the unperturbed system. This puts a restriction on the movement of eigenvalues when the parameter is changed. For this reason, one can expect universal properties only for small parameter differences.

We choose first a in (5) as a parameter of the system then the two densities in (37) differ only in the diffraction coefficient \mathcal{D} . Consequently, the results for the parametric form factor can be obtained directly from the spectral form factor without further calculations. One has to express the contributions to the form factor, (16) and (36), in terms of \mathcal{D}_1 and \mathcal{D}_1^* and replace \mathcal{D}_1^* by \mathcal{D}_2^* . For $\beta = 2$ one obtains in this way

$$\begin{aligned} \tilde{K}_{\text{sc}}(\tau, x) - \tilde{K}_{\text{sc}}(\tau, 0) &\approx \frac{\tau^2}{4} [\mathcal{D}_1 \mathcal{D}_2^* - 2i\mathcal{D}_1 + 2i\mathcal{D}_2^*] \\ &+ \frac{\tau^3}{32} [(\mathcal{D}_1 \mathcal{D}_2^*)^2 - 4i\mathcal{D}_1 \mathcal{D}_2^* (\mathcal{D}_1 - \mathcal{D}_2^*) - 4\mathcal{D}_1 \mathcal{D}_1 - 4\mathcal{D}_2^* \mathcal{D}_2^* + 8\mathcal{D}_1 \mathcal{D}_2^*] \\ &= \frac{\tau^2}{4} [\mathcal{D}_1 \mathcal{D}_2^* - 2i\mathcal{D}_1 + 2i\mathcal{D}_2^*] + \frac{\tau^3}{32} [\mathcal{D}_1 \mathcal{D}_2^* - 2i\mathcal{D}_1 + 2i\mathcal{D}_2^*]^2. \end{aligned} \quad (40)$$

A first point to note is that the parametric form factor in (40) is, in general, not real. This seems to be in contrast with the random matrix result (38) which is real. The reason for this lies, however, in the correct choice of the unfolding procedure. The definition (37) yields only the universal form factor in the case where the mean density of states $\bar{d}(E)$ does not depend on the parameter of the system. However, in the present case the mean density changes slightly with the parameter x of the system, and this leads to a slight shift of the spectrum with x [19]. Consequently, the argument η of the two-level correlation function is shifted, and its Fourier transform, the form factor, is multiplied by a term of the form $e^{ic\tau}$, where c is determined by

the shift of the levels. By rewriting (40) up to the considered order in τ in the form

$$\begin{aligned} \tilde{K}_{sc}(\tau, x) - \tilde{K}_{sc}(\tau, 0) &\approx \left(\tau + \frac{\tau^2}{4} \operatorname{Re} [\mathcal{D}_1 \mathcal{D}_2^* - 2i\mathcal{D}_1 + 2i\mathcal{D}_2^*] \right. \\ &\quad \left. + \frac{\tau^3}{32} (\operatorname{Re} [\mathcal{D}_1 \mathcal{D}_2^* - 2i\mathcal{D}_1 + 2i\mathcal{D}_2^*])^2 \right) \exp \left(\frac{i\tau}{4} \operatorname{Im} [\mathcal{D}_1 \mathcal{D}_2^* - 2i\mathcal{D}_1 + 2i\mathcal{D}_2^*] \right) - \tau \end{aligned} \tag{41}$$

one can extract the result that corresponds to a proper unfolding by dropping the exponential, and one obtains

$$\begin{aligned} K_{sc}(\tau, x) - K_{sc}(\tau, 0) &\approx \frac{\tau^2}{4} \operatorname{Re} [\mathcal{D}_1 \mathcal{D}_2^* - 2i\mathcal{D}_1 + 2i\mathcal{D}_2^*] + \frac{\tau^3}{32} (\operatorname{Re} [\mathcal{D}_1 \mathcal{D}_2^* - 2i\mathcal{D}_1 + 2i\mathcal{D}_2^*])^2 \\ &= -\frac{\tau^2}{8} |\mathcal{D}_1 - \mathcal{D}_2|^2 + \frac{\tau^3}{128} |\mathcal{D}_1 - \mathcal{D}_2|^4 \end{aligned} \tag{42}$$

where (6) has been used.

A comparison with (39) shows that it now has the same form as the random matrix result. The remaining step is to connect the parameter a with the universal parameter x . Since we consider small parameter differences, x is given by $x = \Delta a \sigma_v$, where σ_v is the square root of the variance of the velocities with respect to the parameter a . In order to evaluate σ_v we employ a semiclassical method [20–22] which expresses it in the form

$$\sigma_v^2 = \left\langle \left(\frac{\partial \varepsilon_n}{\partial a} \right)^2 \right\rangle_E = \left\langle \lim_{\eta \rightarrow 0} 2\pi\eta [d_v^\eta(\varepsilon)]^2 \right\rangle_E \tag{43}$$

where d_v^η is a Lorentzian smoothed density of states, that is weighted by the velocities and expressed semiclassically by

$$d_v^\eta(\varepsilon) = \sum_n \frac{\partial \varepsilon_n}{\partial a} \frac{1}{\pi} \frac{\eta}{(\varepsilon - \varepsilon_n)^2 + \eta^2} \approx \operatorname{Im} \frac{\partial}{\partial a} \sum_\gamma \frac{A_\gamma}{\pi T_\gamma} \exp \left\{ \frac{i}{\hbar} S_\gamma - \frac{\eta T_\gamma}{\hbar \bar{d}(E)} \right\}. \tag{44}$$

Here η is the width of the Lorentzian and all other quantities are defined as before.

For small parameter differences we can express the derivative of the diffraction coefficient with respect to a by $(\mathcal{D}_2 - \mathcal{D}_1)/\Delta a$. After inserting (44) into (43) the leading-order contribution comes from single-diffractive orbits, and we evaluate the double sum in the diagonal approximation with the sum rule (13) and amplitudes (10)

$$\begin{aligned} \sigma_v^2 &= \lim_{\eta \rightarrow 0} \int_0^\infty dT \sum_\gamma \frac{\eta \hbar \delta(T - T_\gamma)}{4\pi^2 p_v^2 \beta |M_{12}|} \frac{|\mathcal{D}_2 - \mathcal{D}_1|^2}{(\Delta a)^2} \exp \left\{ -\frac{2\eta T_\gamma}{\hbar \bar{d}(E)} \right\} \\ &= \lim_{\eta \rightarrow 0} \int_0^\infty dT \frac{\eta \hbar}{\beta \Sigma(E)} \frac{|\mathcal{D}_2 - \mathcal{D}_1|^2}{(\Delta a)^2} \exp \left\{ -\frac{2\eta T}{\hbar \bar{d}(E)} \right\} \\ &= \frac{2}{\beta} \frac{|\mathcal{D}_2 - \mathcal{D}_1|^2}{(4\pi \Delta a)^2}. \end{aligned} \tag{45}$$

For $\beta = 2$, the universal parameter is given by $x = \Delta a \sigma_v = |\mathcal{D}_2 - \mathcal{D}_1|/(4\pi)$, and after insertion into (42) we reproduce the random matrix result (39)

$$K_{sc}(\tau, x) - K_{sc}(\tau, 0) \approx -2\pi^2 x^2 \tau^2 + 2\pi^4 x^4 \tau^3. \tag{46}$$

For the GOE ensemble, the parametric density correlation function is given by a triple integral which cannot be expressed in closed form [18]. We consider here only the first-order correction for which the GOE result can be obtained from the asymptotic form of the correlation function for long-range correlations in [3, 23]: $K^{\text{GOE}}(\tau, x) - K^{\text{GOE}}(\tau, 0) \approx -2\pi^2 x^2 \tau^2$. Again we find an agreement with the semiclassical result $K_{sc}(\tau, x) - K_{sc}(\tau, 0) \approx -\tau^2 |\mathcal{D}_1 - \mathcal{D}_2|^2/4 = -2\pi^2 x^2 \tau^2$.

6. Discussion

We have investigated in this paper the influence of a point-like scatterer on the spectral statistics of quantum systems with a chaotic classical limit. It has been shown that the modification of the form factor $K(\tau)$ due to the scatterer can be evaluated systematically in a semiclassical approximation. The expansion of the form factor in powers of τ corresponds on the semiclassical side to an expansion in the number of loops of the diffractive orbits. We have calculated off-diagonal contributions to the τ^2 and τ^3 terms, but the method can be extended to higher-order terms.

The results lead to the conclusion that the delta-perturbation does not modify the form factor. Off-diagonal terms from pairs of different diffractive orbits and from pairs of diffractive and periodic orbits cancel exactly the diagonal terms from diffractive orbits. This requires the existence of correlations between different orbits. These correlations arise from pairs of orbits which are very close in coordinate space.

The results provide a support for the random-matrix conjecture. They imply, up to the considered order, that the statistics of chaotic systems are invariant under the perturbation by a point-like scatterer as is expected from universality. They also show that correlations between two energy spectra for different parameter values are universal, provided that the parameter difference is small. Furthermore, they indicate indirectly that the spectral statistics of the unperturbed system (and thus also of the perturbed system) are identical with those of random matrix theory. The reason for this is that independent results on the invariance of spectral statistics under a delta-perturbation are based on the assumption that the unperturbed energy levels and wavefunctions have random matrix distributions [24, 25]. Since the semiclassical results show this invariance for chaotic systems, the combination of both results provides a theoretical indication that chaotic systems follow the random matrix hypothesis.

Finally, the results are a support for the semiclassical method. They show that semiclassical approximations are capable going beyond the leading term in τ and are an appropriate tool for investigating spectral statistics in the semiclassical limit.

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Appendix. Stability matrix for the motion in a magnetic field

The stability matrix \tilde{M} of a trajectory determines infinitesimal orthogonal deviations from the final point of the trajectory in terms of the deviations from the initial point of the trajectory

$$\begin{pmatrix} dr_f \\ dp_f \end{pmatrix} = \tilde{M} \begin{pmatrix} dr_i \\ dp_i \end{pmatrix}. \quad (\text{A1})$$

In systems with a magnetic field the momentum has the form $\mathbf{p} = m\mathbf{v} + \frac{q}{c}\mathbf{A}(\mathbf{r})$, where q is the charge of the particle and \mathbf{A} is the vector potential. In this case it is often more convenient to consider a matrix M that describes deviations of the velocities instead of those of the momenta. The relation between both matrices is given by

$$\begin{pmatrix} dr_f \\ mdv_f \end{pmatrix} = M \begin{pmatrix} dr_i \\ mdv_i \end{pmatrix} \quad M = A_f^{-1} \tilde{M} A_i \quad A_{i,f} = \begin{pmatrix} 1 & 0 \\ a_{i,f} & 1 \end{pmatrix} \quad (\text{A2})$$

where $a_{i,f} = \frac{q}{c}(\hat{n}_{i,f} \cdot \nabla)(\hat{n}_{i,f} \cdot \mathbf{A}(\mathbf{r}_{i,f}))$, and $\hat{n}_{i,f}$ is the direction orthogonal to the trajectory at the initial and final point of the trajectory, respectively. The matrix M has unit determinant and satisfies $M_{12} = \tilde{M}_{12}$ and $\text{Tr } M = \text{Tr } \tilde{M} + \tilde{M}_{12}(a_i - a_f)$. In cases where the initial and final points are identical and the initial and final velocities differ by a small angle ε , the traces are identical in leading order of ε . Since the semiclassical approximations in this paper involve only \tilde{M}_{12} and $\text{Tr } \tilde{M} - 2$, we express all quantities in terms of M instead of \tilde{M} , and we also use the term stability matrix for it.

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